Energy-momentum and angular momentum densities in gauge theories of gravity

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In the $\overline{\text{Poincaré}}$ gauge theory of gravity, which has been formulated on the basis of a principal fiber bundle over the space-time manifold having the covering group of the proper orthochronous Poincaré group as the structure group, we examine the tensorial properties of the dynamical energy-momentum density ${}^{G}\mathbf{T}_{k}{}^{\mu}$ and the ''spin'' angular momentum density ${}^{G}\mathbf{S}_{kl}{}^{\mu}$ of the gravitational field. They are both space-time vector densities, and transform as tensors under *global* $SL(2,C)$ transformations. Under *local* internal translation, ${}^{G}\mathbf{T}_{k}{}^{\mu}$ is invariant, while ${}^{G}\mathbf{S}_{kl}{}^{\mu}$ transforms inhomogeneously. The dynamical energy-momentum density ${}^{M}\mathbf{T}_{k}{}^{\mu}$ and the ''spin'' angular momentum density ${}^{M}\mathbf{S}_{kl}{}^{\mu}$ of the matter field are also examined, and they are known to be space-time vector densities and to obey tensorial transformation rules under internal $\overline{\text{Poincaré}}$ gauge transformations. The corresponding discussions in extended new general relativity which is obtained as a teleparallel limit of $\overline{\text{Poincaré}}$ gauge theory are also given, and energy-momentum and ''spin'' angular momentum densities are known to be well behaved. Namely, they are all space-time vector densities, etc. The tensorial properties of canonical energy-momentum and ''extended orbital angular momentum'' densities are also examined.

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I. INTRODUCTION

The energy-momentum and angular momentum play central roles in modern theoretical physics. The conservation of these is related to the homogeneity and isotropy of space-time, respectively. Also, local objects such as energy-momentum and angular momentum densities are well defined if the gravitational field does not take part.

In general relativity, however, the energy-momentum and angular momentum densities of the gravitational field so far proposed are not space-time tensor densities. Rather, it is usually asserted [1] that well-behaved energy-momentum and angular momentum densities cannot be defined for the gravitational field, while total energy-momentum and total angular momentum are defined well for asymptotically flat space-time.

In the $\overline{\text{Poincaré}}$ gauge theory of gravity ($\overline{\text{PGT}}$) [2], which has been formulated on the basis of principal fiber bundle over the space-time manifold having the covering group of the proper orthochronous Poincaré group as the structure group, we have defined dynamical energy-momentum and ''spin'' angular momentum densities. For the asymptotically flat space-time, the integration of the dynamical energy-momentum density over spacelike surface σ is the generator of *internal* translation and gives the total energy-momentum of the system. Also, the integration of ''spin'' angular momentum density over σ is the generator of *internal* $SL(2,C)$ transformations and gives the *total* (=*spin*+*orbital*) angular momentum [3], when the Higgs-type field ψ^{k} is chosen to be $\psi^{k}=e^{(0)k}{}_{\mu}x^{\mu}+\psi^{(0)k}+O(1/r^{\beta})$ with constants $e^{(0)k}{}_{\mu},\psi^{(0)k}$ [4–6]. In extended new general relativity (ENGR) which is obtained as a teleparallel limit of PGT, corresponding results have been obtained [7].

The purpose of this paper is to examine, both in $\overline{\text{PGT}}$ and in ENGR, the transformation properties of the dynamical energy-momentum densities, ''spin'' angular momentum densities, canonical energy-momentum densities, and ''extended orbital angular momentum'' densities under general coordinate transformations and under $\overline{\text{Poincaré}}$ gauge transformations [8]. The main result is that *all the dynamical energy-momentum densities and ''spin'' angular momentum densities in these theories are true space-time vector densities.*

II. $\overline{\text{POINCARÉ}}$ GAUGE THEORY

A. Outline of the theory

$\overline{\text{PGT}}$ is formulated on the basis of the principal fiber bundle $\mathcal{P}$ over the space-time M possessing the covering group $\overline{P}_0$ of the proper orthochronous Poincaré group as the structure group. The space-time is assumed to be a noncompact four-dimensional differentiable manifold having a countable base. The bundle $\mathcal{P}$ admits a connection Γ, whose translational and rotational parts of the coefficients will be written as $A^{k}{}_{\mu}$ and $A^{k}{}_{l\mu}$, respectively. The fundamental field variables are $A^{k}{}_{\mu}$, $A^{k}{}_{l\mu}$, the Higgs-type field $\psi=\{\psi^{k}\}$, and the matter field $\phi=\{\phi^{A}|A=1,2,3,\ldots,N\}$ [9]. These fields transform according to [10]

$$\psi'^{k}=(\Lambda(a^{-1}))^{k}{}_{l}(\psi^{l}-t^{l}),$$

$$A'^{k}{}_{\mu}=(\Lambda(a^{-1}))^{k}{}_{l}(A^{l}{}_{\mu}+t^{l}{}_{,\mu}+A^{l}{}_{m\mu}t^{m}),$$

$$A'^{k}{}_{l\mu}=(\Lambda(a^{-1}))^{k}{}_{m}A^{m}{}_{n\mu}(\Lambda(a))^{n}{}_{l}+(\Lambda(a^{-1}))^{k}{}_{m}(\Lambda(a))^{m}{}_{l,\mu},$$

$$\phi'^{A}=[\rho((t,a)^{-1})]^{A}{}_{B}\phi^{B}, \tag{2.1}$$

under the $\overline{\text{Poincaré}}$ gauge transformation

$$\sigma'(x)=\sigma(x)\cdot[t(x),a(x)],$$

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$$t(x)\in T^4,\quad a(x)\in SL(2,C). \tag{2.2}$$

Here, Λ is the covering map from $SL(2,C)$ to the proper orthochronous Lorentz group, and ρ stands for the representation of the $\overline{\text{Poincaré}}$ group to which the field ϕ^A belongs. Also, σ and σ' stand for local cross sections of $\mathcal{P}$. Dual components $e^k{}_\mu$ of vierbeins $e^\mu{}_k\partial/\partial x^\mu$ are related to the field ψ^k and the gauge potentials $A^k{}_\mu$ and $A^k{}_{l\mu}$ through the relation

$$e^k{}_\mu=\psi^k{}_{,\mu}+A^k{}_{l\mu}\psi^l+A^k{}_\mu, \tag{2.3}$$

and these transform according to

$$e'^k{}_\mu=(\Lambda(a^{-1}))^k{}_l e^l{}_\mu, \tag{2.4}$$

under the transformation (2.2). Also, they are related to the metric $g_{\mu\nu}dx^\mu\otimes dx^\nu$ of M through the relation

$$g_{\mu\nu}=\eta_{kl}e^k{}_\mu e^l{}_\nu \tag{2.5}$$

with $(\eta_{kl})\overset{\text{def}}{=}\text{diag}(-1,1,1,1)$.

There is a 2 to 1 bundle homomorphism F from $\mathcal{P}$ to affine frame bundle $\mathcal{A}(M)$ over M, and an extended spinor structure and a spinor structure exist associated with it [12]. The space-time M is orientable, which follows from its assumed noncompactness and from the fact that M has a spinor structure.

The affine frame bundle $\mathcal{A}(M)$ admits a connection Γ_A. The T^4 part $\Gamma^\mu{}_\nu$ and $GL(4,R)$ part $\Gamma^\mu{}_{\nu\lambda}$ of its connection coefficients are related to $A^k{}_{l\mu}$ and $e^k{}_\mu$ through the relations

$$\Gamma^\mu{}_\nu=\delta^\mu{}_\nu,\quad A^k{}_{l\mu}=e^k{}_\nu e^\lambda{}_l\Gamma^\nu{}_{\lambda\mu}+e^k{}_\nu e^\nu{}_{l,\mu}, \tag{2.6}$$

by the requirement that F maps the connection Γ into Γ_A, and the space-time M is of the Riemann-Cartan type.

The field strengths $R^k{}_{l\mu\nu}$, $R^k{}_{\mu\nu}$ and $T^k{}_{\mu\nu}$ of $A^k{}_{l\mu}$, $A^k{}_\mu$ and of $e^k{}_\mu$ are given by [11]

$$R^k{}_{l\mu\nu}\overset{\text{def}}{=}2(A^k{}_{l[\nu,\mu]}+A^k{}_{m[\mu}A^m{}_{l\nu]}),$$

$$R^k{}_{\mu\nu}\overset{\text{def}}{=}2(A^k{}_{[\nu,\mu]}+A^k{}_{l[\mu}A^l{}_{\nu]}),$$

$$T^k{}_{\mu\nu}\overset{\text{def}}{=}2(e^k{}_{[\nu,\mu]}+A^k{}_{l[\mu}e^l{}_{\nu]}), \tag{2.7}$$

and we have the relation

$$T^k{}_{\mu\nu}=R^k{}_{\mu\nu}+R^k{}_{l\mu\nu}\psi^l. \tag{2.8}$$

The field strengths $T^k{}_{\mu\nu}$ and $R^{kl}{}_{\mu\nu}$ are both invariant under *internal* translations. The torsion is given by

$$T^\lambda{}_{\mu\nu}=2\Gamma^\lambda{}_{[\nu\mu]}, \tag{2.9}$$

and the T^4 and $GL(4,R)$ parts of the curvature are given by

$$R^\lambda{}_{\mu\nu}=2(\Gamma^\lambda{}_{[\nu,\mu]}+\Gamma^\lambda{}_{\rho[\mu}\Gamma^\rho{}_{\nu]}), \tag{2.10}$$

$$R^\lambda{}_{\rho\mu\nu}=2(\Gamma^\lambda{}_{\rho[\nu,\mu]}+\Gamma^\lambda{}_{\tau[\mu}\Gamma^\tau{}_{\rho\nu]}), \tag{2.11}$$

respectively. Also, we have

$$T^k{}_{\mu\nu}=e^k{}_\lambda T^\lambda{}_{\mu\nu}=e^k{}_\lambda R^\lambda{}_{\mu\nu}, \tag{2.12}$$

$$R^k{}_{l\mu\nu}=e^k{}_\lambda e^\rho{}_l R^\lambda{}_{\rho\mu\nu}, \tag{2.13}$$

which follow from Eq. (2.6).

The covariant derivative of the matter field ϕ takes the form

$$D_k\phi^A=e^\mu{}_k D_\mu\phi^A,$$

$$D_\mu\phi^A\overset{\text{def}}{=}\partial_\mu\phi^A+\frac{i}{2}A^{lm}{}_\mu(M_{lm}\phi)^A+iA^l{}_\mu(P_l\phi)^A. \tag{2.14}$$

Here, M_{kl} and P_k are representation matrices of the standard basis of the group $\overline{P_0}$: $M_{kl}=-i\rho_*(\overline{M}_{kl})$, $P_k=-i\rho_*(\overline{P}_k)$. The matrix P_k represents the intrinsic energy-momentum of the field ϕ^A [12], and it is vanishing for all the observed fields.

The Lagrangian density [13]

$$\overline{L}=L^M(e^k{}_\mu,\psi^k,D_k\phi^A,\phi^A)+\overline{L^G}(T^{klm},R^{klmn}) \tag{2.15}$$

satisfies the requirement of $\overline{P_0}$ gauge invariance. Here, L^M is the Lagrangian density of the matter field $\phi=\{\phi^A\}$ and $\overline{L^G}$ is the Lagrangian density of the gauge potentials given by

$$\overline{L^G}\overset{\text{def}}{=}L^T+L^R+dR \tag{2.16}$$

with

$$L^T\overset{\text{def}}{=}c_1t^{klm}t_{klm}+c_2v^kv_k+c_3a^ka_k, \tag{2.17}$$

$$L^R\overset{\text{def}}{=}d_1A^{klmn}A_{klmn}+d_2B^{klmn}B_{klmn}+d_3C^{klmn}C_{klmn}+d_4E^{kl}E_{kl}+d_5I^{kl}I_{kl}+d_6R^2. \tag{2.18}$$

In the above, c_i,d_j $(i=1,2,3,j=1,2,3,\dots,6)$ and d are all real constants, t_{klm},v_k and a_k are the irreducible components of the field strength T_{klm}, and $A_{klmn},B_{klmn},C_{klmn},E_{kl},I_{kl}$, and R are the irreducible components of the field strength R_{klmn}. Their definitions are enumerated in the Appendix.

The gravitational Lagrangian density $\overline{L^G}$ agrees with that in Poincaré gauge theory (PGT) [14], and hence *gravitational field equations in* PGT *take the same forms as those in* PGT [15].

In Refs. [4–6], we have used in place of $\overline{L^G}$,

$$L^G\overset{\text{def}}{=}\overline{L^G}+\mathbf{\Delta}/\sqrt{-g}, \tag{2.19}$$

where we have defined

$$g\overset{\text{def}}{=}\det(g_{\mu\nu}),\quad \mathbf{\Delta}\overset{\text{def}}{=}(\mathbf{W}_{kl}{}^{\mu\nu}A^{kl}{}_\mu)_{,\nu} \tag{2.20}$$

with

$$\mathbf{W}_{kl}{}^{\mu\nu} \overset{\text{def}}{=} 2d\sqrt{-g}\,e^{\mu}{}_{[k}e^{\nu}{}_{l]}\,. \tag{2.21}$$

In order to get conserved generators, the Lagrangian density $L \overset{\text{def}}{=} L^G + L^M$, which leads to the same field equations as $\overline{L}$ does, has been employed.

Let us consider Poincaré gauge transformations (2.2) with infinitesimal functions t^k and with $a \in SL(2,C)$ such that $\Lambda(a)$ are represented as

$$(\Lambda(a))^k{}_l = \delta^k{}_l + \omega^k{}_l \tag{2.22}$$

with infinitesimal functions $\omega_{kl} = -\omega_{lk}$. Also, we consider the infinitesimal coordinate transformations

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu} \tag{2.23}$$

with ϵ^{μ} being infinitesimal functions. Under the product transformations of Eqs. (2.22) and (2.23), the fundamental fields $\psi^k, A^k{}_{\mu}, A^{kl}{}_{\mu}$, and ϕ^A and the Lagrangian density L transform according to

$$\psi'^k = \psi^k - \omega^k{}_l \psi^l - t^k,$$

$$A'^k{}_{\mu} = A^k{}_{\mu} - \omega^k{}_l A^l{}_{\mu} + t^k{}_{,\mu} + A^k{}_{l\mu} t^l - \epsilon^{\nu}{}_{,\mu} A^k{}_{\nu}\,,$$

$$A'^{kl}{}_{\mu} = A^{kl}{}_{\mu} + \omega^{kl}{}_{,\mu} - \omega^k{}_m A^{ml}{}_{\mu} - \omega^l{}_m A^{km}{}_{\mu} - \epsilon^{\nu}{}_{,\mu} A^{kl}{}_{\nu}\,,$$

$$\phi'^A = \phi^A - i t^k (P_k \phi)^A - \frac{i}{2}\omega^{kl}(M_{kl}\phi)^A, \tag{2.24}$$

$$L' = L + \frac{1}{\sqrt{-g}}\partial_{\nu}\Lambda^{\nu} \tag{2.25}$$

with

$$\Lambda^{\nu} \overset{\text{def}}{=} \omega^{kl}{}_{,\mu}\mathbf{W}_{kl}{}^{\mu\nu}. \tag{2.26}$$

We see that L is invariant under the product transformations of Eq. (2.22) with const ω_{kl} and Eq. (2.23), but it violates *local* $SL(2,C)$ invariance.

In considering energy-momentum and angular momentum, there are two possibilities in choosing the set of independent field variables [4–6], one is to choose the set $\{\psi^k, A^k{}_{\mu}, A^{kl}{}_{\mu}, \phi^A\}$ and the other is to choose the set $\{\psi^k, e^k{}_{\mu}, A^{kl}{}_{\mu}, \phi^A\}$ instead. In the rest of this section, we employ $\{\psi^k, A^k{}_{\mu}, A^{kl}{}_{\mu}, \phi^A\}$ as the set of independent field variables, because this choice is preferential to the other, as we have seen in Refs. [4–6]. The case when $\{\psi^k, e^k{}_{\mu}, A^{kl}{}_{\mu}, \phi^A\}$ is employed will be mentioned in the final section.

From the transformation properties (2.24) and (2.25), the identities [2]

$$\frac{\delta \mathbf{L}}{\delta \psi^k} + \frac{\delta \mathbf{L}}{\delta A^l{}_{\mu}} A_k{}^l{}_{\mu} + \left(\frac{\delta \mathbf{L}}{\delta A^k{}_{\mu}}\right)_{,\mu} + i\frac{\delta \mathbf{L}}{\delta \phi^A}(P_k\phi)^A \equiv 0, \tag{2.27}$$

$$\frac{\delta \mathbf{L}}{\delta \psi^{[k}}\psi_{l]} + \frac{\delta \mathbf{L}}{\delta A^{[k}{}_{\mu}} A_{l]\mu} + \left(\frac{\delta \mathbf{L}}{\delta A^{kl}{}_{\mu}}\right)_{,\mu} + 2\frac{\delta \mathbf{L}}{\delta A^{[km}{}_{\mu}} A_{l]}{}^m{}_{\mu} + \frac{i}{2}\frac{\delta \mathbf{L}}{\delta \phi^A}(M_{kl}\phi)^A \equiv 0, \tag{2.28}$$

$${}^{\text{tot}}\mathbf{T}_k{}^{\mu} - \mathbf{F}_k{}^{\mu\nu}{}_{,\nu} - \frac{\delta \mathbf{L}}{\delta A^k{}_{\mu}} \equiv 0, \tag{2.29}$$

$$\left({}^{\text{tot}}\mathbf{T}_k{}^{\mu} - \frac{\delta \mathbf{L}}{\delta A^k{}_{\mu}}\right)_{,\mu} \equiv 0, \tag{2.30}$$

$${}^{\text{tot}}\mathbf{S}_{kl}{}^{\mu} + 2\frac{\delta \mathbf{L}}{\delta A^{kl}{}_{\mu}} - \mathbf{\Sigma}_{kl}{}^{\mu\nu}{}_{,\nu} \equiv 0, \tag{2.31}$$

$$\left({}^{\text{tot}}\mathbf{S}_{kl}{}^{\mu} + 2\frac{\delta \mathbf{L}}{\delta A^{kl}{}_{\mu}}\right)_{,\mu} \equiv 0, \tag{2.32}$$

$$\tilde{\mathbf{T}}_{\mu}{}^{\nu} - \partial_{\lambda}\mathbf{\Psi}_{\mu}{}^{\nu\lambda} - \frac{\delta \mathbf{L}}{\delta A^k{}_{\nu}} A^k{}_{\mu} - \frac{\delta \mathbf{L}}{\delta A^{kl}{}_{\nu}} A^{kl}{}_{\mu} \equiv 0 \tag{2.33}$$

follow, where we have defined

$$\mathbf{L} \overset{\text{def}}{=} \sqrt{-g}L, \tag{2.34}$$

$$\mathbf{F}_k{}^{\mu\nu} \overset{\text{def}}{=} \frac{\partial \mathbf{L}}{\partial A^k{}_{\mu,\nu}} = \frac{\partial \mathbf{L}^G}{\partial A^k{}_{\mu,\nu}} = -\mathbf{F}_k{}^{\nu\mu}, \tag{2.35}$$

$${}^{\text{tot}}\mathbf{T}_k{}^{\mu} \overset{\text{def}}{=} \mathbf{F}_k{}^{\mu} + i\frac{\partial \mathbf{L}}{\partial \phi^A{}_{,\mu}}(P_k\phi)^A + \mathbf{F}_l{}^{\nu\mu} A_k{}^l{}_{\nu}\,, \tag{2.36}$$

$${}^{\text{tot}}\mathbf{S}_{kl}{}^{\mu} \overset{\text{def}}{=} -2\mathbf{F}_{[k}{}^{\mu}\psi_{l]} - 2\mathbf{F}_{[k}{}^{\nu\mu}A_{l]\nu} - 4\mathbf{F}_{[km}{}^{\nu\mu}A_{l]}{}^m{}_{\nu} - i\frac{\partial \mathbf{L}}{\partial \phi^A{}_{,\mu}}(M_{kl}\phi)^A, \tag{2.37}$$

$$\tilde{\mathbf{T}}_{\mu}{}^{\nu} \overset{\text{def}}{=} \delta_{\mu}{}^{\nu}\mathbf{L} - \mathbf{F}_k{}^{\lambda\nu}A^k{}_{\lambda,\mu} - \mathbf{F}_{kl}{}^{\lambda\nu}A^{kl}{}_{\lambda,\mu} - \mathbf{F}_k{}^{\nu}\psi^k{}_{,\mu} - \frac{\partial \mathbf{L}}{\partial \phi^A{}_{,\nu}}\phi^A{}_{,\mu}\,, \tag{2.38}$$

$$\mathbf{\Sigma}_{kl}{}^{\mu\nu} \overset{\text{def}}{=} -2\mathbf{F}_{kl}{}^{\mu\nu} + 2\mathbf{W}_{kl}{}^{\mu\nu} = -\Sigma_{kl}{}^{\nu\mu} \tag{2.39}$$

with

$$\mathbf{F}_k{}^{\mu} \stackrel{\text{def}}{=} \frac{\partial \mathbf{L}}{\partial \psi^k{}_{,\mu}}, \tag{2.40}$$

$$\mathbf{F}_{kl}{}^{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial \mathbf{L}}{\partial A^{kl}{}_{\mu,\nu}} = \frac{\partial \mathbf{L}^G}{\partial A^{kl}{}_{\mu,\nu}} = -\mathbf{F}_{kl}{}^{\nu\mu}, \tag{2.41}$$

$$\mathbf{\Psi}_\lambda{}^{\mu\nu} \stackrel{\text{def}}{=} \mathbf{F}_k{}^{\mu\nu} A^k{}_\lambda + \mathbf{F}_{kl}{}^{\mu\nu} A^{kl}{}_\lambda = -\mathbf{\Psi}_\lambda{}^{\nu\mu}. \tag{2.42}$$

The energy-momentum density ${}^{\text{tot}}\mathbf{T}_k{}^{\mu}$ and the ''spin'' angular momentum density ${}^{\text{tot}}\mathbf{S}_{kl}{}^{\mu}$ are expressed as follows:

$${}^{\text{tot}}\mathbf{T}_k{}^{\mu} = \frac{\partial \mathbf{L}}{\partial A^k{}_\mu}, \tag{2.43}$$

$${}^{\text{tot}}\mathbf{S}_{kl}{}^{\mu} = -2\frac{\partial \mathbf{L}}{\partial A^{kl}{}_\mu} + 2\mathbf{W}_{kl}{}^{\mu\nu}{}_{,\nu}, \tag{2.44}$$

by virtue of the identities (2.29) and (2.31). Thus, ${}^{\text{tot}}\mathbf{T}_k{}^{\mu}$ has the standard form of gauge current in Yang-Mills theories, while ${}^{\text{tot}}\mathbf{S}_{kl}{}^{\mu}$ has not. There is an additional term $2\mathbf{W}_{kl}{}^{\mu\nu}{}_{,\nu}$ which originates from the term $\mathbf{\Delta}$ violating the $SL(2,C)$-gauge invariance of the gravitational Lagrangian density.

When the field equations $\delta \mathbf{L}/\delta A^k{}_\mu \stackrel{\text{def}}{=} \partial \mathbf{L}/\partial A^k{}_\mu - \partial_\nu(\partial \mathbf{L}/\partial A^k{}_{\mu,\nu}) = 0$ and $\delta \mathbf{L}/\delta \phi^A = 0$ are both satisfied, we have the following:

(i) The field equation $\delta \mathbf{L}/\delta \psi^k = 0$ is automatically satisfied, and hence ψ^k is not an independent dynamical variable.

(ii)

$$\partial_\mu {}^{\text{tot}}\mathbf{T}_k{}^{\mu} = 0, \tag{2.45}$$

$$\partial_\mu {}^{\text{tot}}\mathbf{S}_{kl}{}^{\mu} = 0, \tag{2.46}$$

which are the differential conservation laws of the dynamical energy-momentum and of the ''spin'' angular momentum, respectively. (i) and (ii) follow from Eqs. (2.27), (2.30), and (2.32).

Equations (2.33) and (2.42) lead to

$$\partial_\nu \tilde{\mathbf{T}}_\mu{}^{\nu} = 0, \tag{2.47}$$

$$\partial_\nu \tilde{\mathbf{M}}_\lambda{}^{\mu\nu} = 0, \tag{2.48}$$

when $\delta \mathbf{L}/\delta A^k{}_\mu = 0$, $\delta \mathbf{L}/\delta A^{kl}{}_\mu = 0$, where $\tilde{\mathbf{M}}_\lambda{}^{\mu\nu} \stackrel{\text{def}}{=} 2(\mathbf{\Psi}_\lambda{}^{\mu\nu} - x^\mu \tilde{\mathbf{T}}_\lambda{}^{\nu})$. Equations (2.47) and (2.48) are the differential conservation laws of the canonical energy-momentum and ''extended orbital angular momentum'' [5,6], respectively.

In Refs. [4–6], we have examined the integrations of ${}^{\text{tot}}\mathbf{T}_k{}^{\mu}$, ${}^{\text{tot}}\mathbf{S}_{kl}{}^{\mu}$, $\tilde{\mathbf{T}}_\mu{}^{\nu}$ and $\tilde{\mathbf{M}}_\lambda{}^{\mu\nu}$ for asymptotically flat spacetime by choosing ψ^k as

$$\psi^k = e^{(0)k}{}_\mu x^\mu + \psi^{(0)k} + O(1/r^\beta),$$

$$\psi^k{}_{,\mu} = e^{(0)k}{}_\mu + O(1/r^{\beta+1}), \quad (\beta > 0), \tag{2.49}$$

where $e^{(0)k}{}_\mu$ is a constant satisfying $e^{(0)k}{}_\mu \eta_{kl} e^{(0)l}{}_\nu = \eta_{\mu\nu}$, and $\psi^{(0)k}$ and β are constants. Also, we have defined $r \stackrel{\text{def}}{=} \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, and $O(1/r^\alpha)$ with positive α denotes a term for which $r^\alpha O(1/r^\alpha)$ remains finite for $r \to \infty$; a term $O(1/r^\alpha)$ may of course be zero. We have shown the following:

$$M_k \stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\mathbf{T}_k{}^{\mu} d\sigma_\mu = e^{(0)\mu}{}_k M_\mu, \tag{2.50}$$

$$S_{kl} \stackrel{\text{def}}{=} \int_\sigma {}^{\text{tot}}\mathbf{S}_{kl}{}^{\mu} d\sigma_\mu = e^{(0)}{}_{k\mu} e^{(0)}{}_{l\nu} M^{\mu\nu} + 2\psi^{(0)}{}_{[k} M_{l]}, \tag{2.51}$$

$$M^c{}_\mu \stackrel{\text{def}}{=} \int_\sigma \tilde{\mathbf{T}}_\mu{}^{\nu} d\sigma_\nu = 0, \tag{2.52}$$

$$L_\mu{}^{\nu} \stackrel{\text{def}}{=} \int_\sigma \tilde{\mathbf{M}}_\mu{}^{\nu\lambda} d\sigma_\lambda = 0, \tag{2.53}$$

where $d\sigma_\mu$ denotes the surface element on a spacelike surface σ. Also, we have defined

$$M_\mu \stackrel{\text{def}}{=} \eta_{\mu\lambda} \int_\sigma \theta^{\lambda\nu} d\sigma_\nu, \tag{2.54}$$

$$M^{\mu\nu} \stackrel{\text{def}}{=} \int_\sigma \partial_\rho K^{\mu\nu\lambda\rho} d\sigma_\lambda = \int_\sigma (x^\mu \theta^{\nu\lambda} - x^\nu \theta^{\mu\lambda}) d\sigma_\lambda \tag{2.55}$$

with

$$\theta^{\lambda\nu} \stackrel{\text{def}}{=} 2d\,\partial_\rho \partial_\sigma \{(-g) g^{\lambda[\nu} g^{\rho]\sigma}\}, \tag{2.56}$$

$$K^{\mu\nu\lambda\rho} \stackrel{\text{def}}{=} 2d[x^\mu \partial_\sigma \{(-g) g^{\nu[\lambda} g^{\rho]\sigma}\} - x^\nu \partial_\sigma \{(-g) g^{\mu[\lambda} g^{\rho]\sigma}\} + (-g) g^{\mu[\lambda} g^{\rho]\nu}]. \tag{2.57}$$

Actually, Eq. (2.50) has been obtained without using Eq. (2.49), but it is crucial in obtaining expressions (2.51)–(2.53). Also, the expression of $\theta^{\lambda\nu}$ agrees with that of the symmetric energy-momentum density proposed by Landau-Lifshitz in general relativity.

The dynamical energy-momentum M_k is the generator of internal translations and the total energy momentum of the system. The ''spin'' angular momentum S_{kl} is the generator of internal $SL(2,C)$ transformations and the *total* (=*spin* +*orbital*) angular momentum of the system. The canonical energy-momentum $M^c{}_\mu$ and the ''extended orbital angular momentum'' $L_\mu{}^{\nu}$ are the generators of coordinate translations and of coordinate $GL(4,R)$ transformations, respectively [6,16].

The following is worth emphasizing: The total energy-momentum and the total angular momentum are the generators of *internal* Poincaré transformations, and the generators of coordinate transformations are vanishing and trivial.

B. Transformation properties of energy-momentum and angular momentum densities

We define densities ${}^{G}\mathbf{T}_k{}^{\mu}$, ${}^{M}\mathbf{T}_k{}^{\mu}$, ${}^{G}\mathbf{S}_{kl}{}^{\mu}$ and ${}^{M}\mathbf{S}_{kl}{}^{\mu}$ by

$$ {}^{G}\mathbf{T}_k{}^{\mu} \stackrel{\mathrm{def}}{=} \frac{\partial \mathbf{L}^G}{\partial \psi^k{}_{,\mu}} + \mathbf{F}_l{}^{\nu\mu} A_k{}^l{}_{\nu} = \frac{\partial \mathbf{L}^G}{\partial A^k{}_{\mu}}, \tag{2.58} $$

$$ {}^{M}\mathbf{T}_k{}^{\mu} \stackrel{\mathrm{def}}{=} \frac{\partial \mathbf{L}^M}{\partial \psi^k{}_{,\mu}} + i\frac{\partial \mathbf{L}^M}{\partial \phi^A{}_{,\mu}} (P_k\phi)^A = \frac{\partial \mathbf{L}^M}{\partial A^k{}_{\mu}}, \tag{2.59} $$

$$ \begin{aligned} {}^{G}\mathbf{S}_{kl}{}^{\mu} &\stackrel{\mathrm{def}}{=} -2\frac{\partial \mathbf{L}^G}{\partial \psi^{[k}{}_{,\mu}} \psi_{l]} - 2\mathbf{F}_{[k}{}^{\nu\mu} A_{l]\nu} - 4\mathbf{F}_{[km}{}^{\nu\mu} A_{l]}{}^m{}_{\nu} \\ &= -2\frac{\partial \mathbf{L}^G}{\partial A^{kl}{}_{\mu}} + 2\mathbf{W}_{kl}{}^{\mu\nu}{}_{,\nu}, \end{aligned} \tag{2.60} $$

$$ \begin{aligned} {}^{M}\mathbf{S}_{kl}{}^{\mu} &\stackrel{\mathrm{def}}{=} -2\frac{\partial \mathbf{L}^M}{\partial \psi^{[k}{}_{,\mu}} \psi_{l]} - i\frac{\partial \mathbf{L}^M}{\partial \phi^A{}_{,\mu}} (M_{kl}\phi)^A \\ &= -2\frac{\partial \mathbf{L}^M}{\partial A^{kl}{}_{\mu}} \end{aligned} \tag{2.61} $$

with $\mathbf{L}^G \stackrel{\mathrm{def}}{=} \sqrt{-g}L^G$ and $\mathbf{L}^M \stackrel{\mathrm{def}}{=} \sqrt{-g}L^M$. The densities ${}^{G}\mathbf{T}_k{}^{\mu}$ and ${}^{M}\mathbf{T}_k{}^{\mu}$ are the dynamical energy-momentum densities of the gravitational field and of the matter field ϕ^A, respectively, while ${}^{G}\mathbf{S}_{kl}{}^{\mu}$ and ${}^{M}\mathbf{S}_{kl}{}^{\mu}$ are ''spin'' angular momentum densities of the gravitational and the matter fields, respectively. There are the relations

$$ {}^{\mathrm{tot}}\mathbf{T}_k{}^{\mu} = {}^{G}\mathbf{T}_k{}^{\mu} + {}^{M}\mathbf{T}_k{}^{\mu}, \quad {}^{\mathrm{tot}}\mathbf{S}_{kl}{}^{\mu} = {}^{G}\mathbf{S}_{kl}{}^{\mu} + {}^{M}\mathbf{S}_{kl}{}^{\mu}. \tag{2.62} $$

Under the product transformations of Eqs. (2.22) and (2.23), the densities ${}^{G}\mathbf{T}_k{}^{\mu}$, ${}^{M}\mathbf{T}_k{}^{\mu}$, ${}^{G}\mathbf{S}_{kl}{}^{\mu}$ and ${}^{M}\mathbf{S}_{kl}{}^{\mu}$ transform according to

$$ \begin{aligned} {}^{G}\mathbf{T}'{}_k{}^{\mu} &= {}^{G}\mathbf{T}_k{}^{\mu} - \omega_k{}^l\, {}^{G}\mathbf{T}_l{}^{\mu} + \epsilon^{\mu}{}_{,\nu}\, {}^{G}\mathbf{T}_k{}^{\nu} - \epsilon^{\lambda}{}_{,\lambda}\, {}^{G}\mathbf{T}_k{}^{\mu} - \omega_k{}^l{}_{,\nu}\mathbf{F}_l{}^{\mu\nu} \\ &\quad + \frac{\partial(\partial_\nu \mathbf{\Lambda}^\nu)}{\partial A^k{}_{\mu}}, \end{aligned} \tag{2.63} $$

$$ {}^{M}\mathbf{T}'{}_k{}^{\mu} = {}^{M}\mathbf{T}_k{}^{\mu} - \omega_k{}^l\, {}^{M}\mathbf{T}_l{}^{\mu} + \epsilon^{\mu}{}_{,\nu}\, {}^{M}\mathbf{T}_k{}^{\nu} - \epsilon^{\lambda}{}_{,\lambda}\, {}^{M}\mathbf{T}_k{}^{\mu}, \tag{2.64} $$

$$ \begin{aligned} {}^{G}\mathbf{S}'{}_{kl}{}^{\mu} &= {}^{G}\mathbf{S}_{kl}{}^{\mu} - \omega_k{}^m\, {}^{G}\mathbf{S}_{ml}{}^{\mu} - \omega_l{}^m\, {}^{G}\mathbf{S}_{km}{}^{\mu} - 2t_{[k}\, {}^{G}\mathbf{T}_{l]}{}^{\mu} \\ &\quad + \epsilon^{\mu}{}_{,\nu}\, {}^{G}\mathbf{S}_{kl}{}^{\nu} - \epsilon^{\lambda}{}_{,\lambda}\, {}^{G}\mathbf{S}_{kl}{}^{\mu} - 2t_{[k,\nu}\mathbf{F}_{l]}{}^{\mu\nu} \\ &\quad - 2\omega_{[k}{}^m{}_{,\nu}\mathbf{\Sigma}_{l]m}{}^{\mu\nu} - 2\frac{\partial(\partial_\nu \mathbf{\Lambda}^\nu)}{\partial A^{kl}{}_{\mu}}, \end{aligned} \tag{2.65} $$

$$ \begin{aligned} {}^{M}\mathbf{S}'{}_{kl}{}^{\mu} &= {}^{M}\mathbf{S}_{kl}{}^{\mu} - \omega_k{}^m\, {}^{M}\mathbf{S}_{ml}{}^{\mu} - \omega_l{}^m\, {}^{M}\mathbf{S}_{km}{}^{\mu} - 2t_{[k}\, {}^{M}\mathbf{T}_{l]}{}^{\mu} \\ &\quad + \epsilon^{\mu}{}_{,\nu}\, {}^{M}\mathbf{S}_{kl}{}^{\nu} - \epsilon^{\lambda}{}_{,\lambda}\, {}^{M}\mathbf{S}_{kl}{}^{\mu}. \end{aligned} \tag{2.66} $$

We define ${}^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu}$, ${}^{M}\mathbf{T}_{\mu}{}^{\nu}$ by

$$ {}^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu} \stackrel{\mathrm{def}}{=} \delta_\mu{}^\nu \mathbf{L}^G - \mathbf{F}_k{}^{\lambda\nu} A^k{}_{\lambda,\mu} - \mathbf{F}_{kl}{}^{\lambda\nu} A^{kl}{}_{\lambda,\mu} - \frac{\partial \mathbf{L}^G}{\partial \psi^k{}_{,\nu}} \psi^k{}_{,\mu}, \tag{2.67} $$

$$ {}^{M}\mathbf{T}_{\mu}{}^{\nu} \stackrel{\mathrm{def}}{=} \delta_\mu{}^\nu \mathbf{L}^M - \frac{\partial \mathbf{L}^M}{\partial \psi^k{}_{,\nu}} \psi^k{}_{,\mu} - \frac{\partial \mathbf{L}^M}{\partial \phi^A{}_{,\nu}} \phi^A{}_{,\mu}, \tag{2.68} $$

which are the canonical energy-momentum densities of the gravitational field and of the matter field, respectively. Also, we define

$$ {}^{G}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu} \stackrel{\mathrm{def}}{=} 2(\mathbf{\Psi}_{\lambda}{}^{\mu\nu} - x^\mu\, {}^{G}\tilde{\mathbf{T}}_{\lambda}{}^{\nu}), \tag{2.69} $$

$$ {}^{M}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu} \stackrel{\mathrm{def}}{=} -2x^\mu\, {}^{M}\mathbf{T}_{\lambda}{}^{\nu}, \tag{2.70} $$

which are the ''extended orbital angular momentum'' densities of the gravitational field and of the matter field, respectively. There are the relations

$$ \tilde{\mathbf{T}}_{\mu}{}^{\nu} = {}^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu} + {}^{M}\mathbf{T}_{\mu}{}^{\nu}, \quad \tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu} = {}^{G}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu} + {}^{M}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu}. \tag{2.71} $$

The densities ${}^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu}$, ${}^{M}\mathbf{T}_{\mu}{}^{\nu}$, ${}^{G}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu}$ and ${}^{M}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu}$ transform according to

$$ \begin{aligned} {}^{G}\tilde{\mathbf{T}}'{}_{\mu}{}^{\nu} &= {}^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu} - (t^k{}_{,\lambda\mu} + A^k{}_{l\lambda}t^l{}_{,\mu})\mathbf{F}_k{}^{\lambda\nu} - (\omega^{kl}{}_{,\lambda\mu} - \omega^k{}_{m,\mu}A^{ml}{}_{\lambda} - \omega^l{}_{m,\mu}A^{km}{}_{\lambda})\mathbf{F}_{kl}{}^{\lambda\nu} + t^k{}_{,\mu}\frac{\partial \mathbf{L}^G}{\partial \psi^k{}_{,\nu}} + \omega^{kl}{}_{,\mu}\frac{\partial \mathbf{L}^G}{\partial \psi^{[k}{}_{,\nu}}\psi_{l]} \\ &\quad + (\partial_\rho \mathbf{\Lambda}^\rho)\delta_\mu{}^\nu - \frac{\partial(\partial_\rho \mathbf{\Lambda}^\rho)}{\partial \psi^k{}_{,\nu}}\psi^k{}_{,\mu} - \frac{\partial(\partial_\rho \mathbf{\Lambda}^\rho)}{\partial A^k{}_{\lambda,\nu}}A^k{}_{\lambda,\mu} - \frac{\partial(\partial_\rho \mathbf{\Lambda}^\rho)}{\partial A^{kl}{}_{\lambda,\nu}}A^{kl}{}_{\lambda,\mu} + \epsilon^{\lambda}{}_{,\mu}\, {}^{G}\tilde{\mathbf{T}}_{\lambda}{}^{\nu} - \epsilon^{\nu}{}_{,\lambda}\, {}^{G}\tilde{\mathbf{T}}_{\mu}{}^{\lambda} - \epsilon^{\lambda}{}_{,\lambda}\, {}^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu} + \epsilon^{\rho}{}_{,\lambda\mu}\mathbf{\Psi}_{\rho}{}^{\lambda\nu}, \end{aligned} \tag{2.72} $$

$$ {}^{M}\mathbf{T}'{}_{\mu}{}^{\nu} = {}^{M}\mathbf{T}_{\mu}{}^{\nu} + t^k{}_{,\mu}\frac{\partial \mathbf{L}^M}{\partial \psi^k{}_{,\nu}} + \omega^{kl}{}_{,\mu}\frac{\partial \mathbf{L}^M}{\partial \psi^{[k}{}_{,\nu}}\psi_{l]} + \epsilon^{\lambda}{}_{,\mu}\, {}^{M}\mathbf{T}_{\lambda}{}^{\nu} - \epsilon^{\nu}{}_{,\lambda}\, {}^{M}\mathbf{T}_{\mu}{}^{\lambda} - \epsilon^{\lambda}{}_{,\lambda}\, {}^{M}\mathbf{T}_{\mu}{}^{\nu}, \tag{2.73} $$

$$ {}^{G}\tilde{\mathbf{M}}'{}_{\lambda}{}^{\mu\nu} = 2(\mathbf{\Psi}'{}_{\lambda}{}^{\mu\nu} - x^\mu\, {}^{G}\tilde{\mathbf{T}}'{}_{\lambda}{}^{\nu} - \epsilon^\mu\, {}^{G}\tilde{\mathbf{T}}_{\lambda}{}^{\nu}), \tag{2.74} $$

$$ {}^{M}\tilde{\mathbf{M}}'{}_{\lambda}{}^{\mu\nu} = -2x^\mu\, {}^{M}\mathbf{T}'{}_{\lambda}{}^{\nu} - 2\epsilon^\mu\, {}^{M}\mathbf{T}_{\lambda}{}^{\nu}, \tag{2.75} $$

under the product transformations of Eqs. (2.22) and (2.23), where $\mathbf{\Psi}'_{\lambda}{}^{\mu\nu}$ denotes the transformed $\mathbf{\Psi}_{\lambda}{}^{\mu\nu}$:

$$\mathbf{\Psi}'_{\lambda}{}^{\mu\nu}=\mathbf{\Psi}_{\lambda}{}^{\mu\nu}+t^{k}{}_{,\lambda}\mathbf{F}_{k}{}^{\mu\nu}+\omega^{kl}{}_{,\lambda}\mathbf{F}_{kl}{}^{\mu\nu}+\frac{\partial(\partial_\rho\Lambda^\rho)}{\partial A^{k}{}_{\mu,\nu}}A^{k}{}_{\lambda}+\frac{\partial(\partial_\rho\Lambda^\rho)}{\partial A^{kl}{}_{\mu,\nu}}A^{kl}{}_{\lambda}-\epsilon^{\rho}{}_{,\lambda}\mathbf{\Psi}_{\rho}{}^{\mu\nu}+\epsilon^{\mu}{}_{,\rho}\mathbf{\Psi}_{\lambda}{}^{\rho\nu}+\epsilon^{\nu}{}_{,\rho}\mathbf{\Psi}_{\lambda}{}^{\mu\rho}-\epsilon^{\rho}{}_{,\rho}\mathbf{\Psi}_{\lambda}{}^{\mu\nu}. \tag{2.76}$$

III. EXTENDED NEW GENERAL RELATIVITY

A. Reduction of $\overline{\text{Poincaré}}$ gauge theory to extended new general relativity

In $\overline{\text{P}}$GT, we consider the case in which the field strength $R^{kl}{}_{\mu\nu}$ vanishes identically,

$$R^{kl}{}_{\mu\nu}\equiv 0, \tag{3.1}$$

then, the curvature vanishes and we have a teleparallel theory.

We choose the $SL(2,C)$ gauge such that

$$A^{kl}{}_{\mu}\equiv 0, \tag{3.2}$$

which reduces the expressions of vierbeins $e^{k}{}_{\mu}$, affine connection coefficients $\Gamma^{\lambda}{}_{\mu\nu}$ and the covariant derivative $D_k\phi$ to

$$e^{k}{}_{\mu}=\psi^{k}{}_{,\mu}+A^{k}{}_{\mu}, \tag{3.3}$$

$$\Gamma^{\lambda}{}_{\mu\nu}=e^{\lambda}{}_{l}e^{l}{}_{\mu,\nu}, \tag{3.4}$$

$$D_k\phi^A=e^{\mu}{}_{k}D_\mu\phi^A, \tag{3.5}$$

$$D_\mu\phi^A=\partial_\mu\phi^A+iA^{l}{}_{\mu}(P_l\phi)^A, \tag{3.6}$$

respectively.

Since $L^R=0=dR$, the gravitational Lagrangian density L^G is reduced to $L^G=L^T$, which agrees with the gravitational Lagrangian density in new general relativity (NGR) [17,18]. Thus, *the gravitational field equations in* ENGR *take the same forms as those in* NGR.

As is the case of the Lagrangian density L^G+L^M in $\overline{\text{P}}$GT, $L=L^T+L^M$ is invariant under the product transformations of Eq. (2.22) with const ω_{kl} and Eq. (2.23), but it violates *local* $SL(2,C)$ invariance.

The identities (2.27) and (2.28) and the definitions (2.36) and (2.37) are reduced to [19]

$$\frac{\delta\mathbf{L}}{\delta\psi^k}+\left(\frac{\delta\mathbf{L}}{\delta A^{k}{}_{\mu}}\right)_{,\mu}+i\frac{\delta\mathbf{L}}{\delta\phi^A}(P_k\phi)^A\equiv 0, \tag{3.7}$$

$$\partial_\mu{}^{\mathrm{tot}}\mathbf{S}_{kl}{}^{\mu}-2\frac{\delta\mathbf{L}}{\delta\psi^{[k}}\psi_{l]}-2\frac{\delta\mathbf{L}}{\delta A^{[k}{}_{\mu}}A_{l]\mu}-i\frac{\delta\mathbf{L}}{\delta\phi^A}(M_{kl}\phi)^A\equiv 0, \tag{3.8}$$

$${}^{\mathrm{tot}}\mathbf{T}_{k}{}^{\mu}=\mathbf{F}_{k}{}^{\mu}+i\frac{\partial\mathbf{L}}{\partial\phi^{A}{}_{,\mu}}(P_k\phi)^A, \tag{3.9}$$

$${}^{\mathrm{tot}}\mathbf{S}_{kl}{}^{\mu}=-2\mathbf{F}_{[k}{}^{\mu}\psi_{l]}-2\mathbf{F}_{[k}{}^{\nu\mu}A_{l]\nu}-i\frac{\partial\mathbf{L}}{\partial\phi^{A}{}_{,\mu}}(M_{kl}\phi)^A, \tag{3.10}$$

respectively. The identities (2.29) and (2.30), the expression (2.43), and the conservation laws (2.45) and (2.46) remain unchanged and there is no identity corresponding to Eqs. (2.31) and (2.32) and no expression corresponding to Eq. (2.44).

The field equation $\delta\mathbf{L}/\delta\psi^k=0$ is automatically satisfied, if the field equations $\delta\mathbf{L}/\delta A^{k}{}_{\mu}=0$ and $\delta\mathbf{L}/\delta\phi^A=0$ are both satisfied.

The identity (2.33), the definitions (2.38) and (2.42) are reduced to

$$\tilde{\mathbf{T}}_{\mu}{}^{\nu}-\partial_\lambda\mathbf{\Psi}_{\mu}{}^{\nu\lambda}-\frac{\delta\mathbf{L}}{\delta A^{k}{}_{\nu}}A^{k}{}_{\mu}\equiv 0, \tag{3.11}$$

$$\tilde{\mathbf{T}}_{\mu}{}^{\nu}\stackrel{\mathrm{def}}{=}\delta_\mu{}^\nu\mathbf{L}-\mathbf{F}_{k}{}^{\lambda\nu}A^{k}{}_{\lambda,\mu}-\frac{\partial\mathbf{L}}{\partial\phi^{A}{}_{,\nu}}\phi^{A}{}_{,\mu}-\mathbf{F}_{k}{}^{\nu}\psi^{k}{}_{,\mu}, \tag{3.12}$$

$$\mathbf{\Psi}_{\lambda}{}^{\mu\nu}\stackrel{\mathrm{def}}{=}\mathbf{F}_{k}{}^{\mu\nu}A^{k}{}_{\lambda}=-\mathbf{\Psi}_{\lambda}{}^{\nu\mu}, \tag{3.13}$$

respectively. The conservation laws (2.47) and (2.48) remain unchanged.

For the case with $c_1=-1/(3\kappa)=-c_2$, we have examined [7] $M_k, S_{kl}, M^{c}{}_{\mu}$, and $L_{\mu}{}^{\nu}$ defined in the same ways as in PGT for asymptotically flat space-time by choosing ψ^k as given by Eq. (2.49) and assuming some additional conditions on asymptotic behaviors of field variables. The same expressions as Eqs. (2.50)–(2.53) hold also for this case.

B. Transformation properties of energy-momentum and angular momentum densities

In the case of ENGR, the energy-momentum and ''spin'' angular momentum densities ${}^{G}\mathbf{T}_{k}{}^{\mu}$, ${}^{M}\mathbf{T}_{k}{}^{\mu}$, ${}^{G}\mathbf{S}_{kl}{}^{\mu}$ and ${}^{M}\mathbf{S}_{kl}{}^{\mu}$ in $\overline{\text{P}}$GT reduce to [20]

$${}^{G}\mathbf{T}_{k}{}^{\mu}=\frac{\partial\mathbf{L}^T}{\partial\psi^{k}{}_{,\mu}}=\frac{\partial\mathbf{L}^T}{\partial A^{k}{}_{\mu}}, \tag{3.14}$$

$${}^{M}\mathbf{T}_{k}{}^{\mu}=\frac{\partial\mathbf{L}^M}{\partial\psi^{k}{}_{,\mu}}+i\frac{\partial\mathbf{L}^M}{\partial\phi^{A}{}_{,\mu}}(P_k\phi)^A=\frac{\partial\mathbf{L}^M}{\partial A^{k}{}_{\mu}}, \tag{3.15}$$

$${}^{G}\mathbf{S}_{kl}{}^{\mu}=-2\frac{\partial\mathbf{L}^T}{\partial\psi^{[k}{}_{,\mu}}\psi_{l]}-2\mathbf{F}_{[k}{}^{\nu\mu}A_{l]\nu}, \tag{3.16}$$

$${}^{M}\mathbf{S}_{kl}{}^{\mu}=-2\frac{\partial\mathbf{L}^M}{\partial\psi^{[k}{}_{,\mu}}\psi_{l]}-i\frac{\partial\mathbf{L}^M}{\partial\phi^{A}{}_{,\mu}}(M_{kl}\phi)^A. \tag{3.17}$$

Under the product transformations of Eqs. (2.22) and (2.23), these densities transform according as

$$^{G}\mathbf{T}'{}_{k}{}^{\mu}={}^{G}\mathbf{T}_{k}{}^{\mu}-\omega_{k}{}^{l}\,{}^{G}\mathbf{T}_{l}{}^{\mu}+\epsilon^{\mu}{}_{,\nu}{}^{G}\mathbf{T}_{k}{}^{\nu}-\epsilon^{\lambda}{}_{,\lambda}{}^{G}\mathbf{T}_{k}{}^{\mu}-\omega_{k}{}^{l}{}_{,\nu}\mathbf{F}_{l}{}^{\mu\nu}, \tag{3.18}$$

$$^{M}\mathbf{T}'{}_{k}{}^{\mu}={}^{M}\mathbf{T}_{k}{}^{\mu}-\omega_{k}{}^{l}\,{}^{M}\mathbf{T}_{l}{}^{\mu}+\epsilon^{\mu}{}_{,\nu}{}^{M}\mathbf{T}_{k}{}^{\nu}-\epsilon^{\lambda}{}_{,\lambda}{}^{M}\mathbf{T}_{k}{}^{\mu}, \tag{3.19}$$

$$^{G}\mathbf{S}'{}_{kl}{}^{\mu}={}^{G}\mathbf{S}_{kl}{}^{\mu}-\omega_{k}{}^{m}\,{}^{G}\mathbf{S}_{ml}{}^{\mu}-\omega_{l}{}^{m}\,{}^{G}\mathbf{S}_{km}{}^{\mu}-2t_{[k}{}^{G}\mathbf{T}_{l]}{}^{\mu}+\epsilon^{\mu}{}_{,\nu}{}^{G}\mathbf{S}_{kl}{}^{\nu}-\epsilon^{\lambda}{}_{,\lambda}{}^{G}\mathbf{S}_{kl}{}^{\mu}-2t_{[k,\nu}\mathbf{F}_{l]}{}^{\mu\nu}, \tag{3.20}$$

$$^{M}\mathbf{S}'{}_{kl}{}^{\mu}={}^{M}\mathbf{S}_{kl}{}^{\mu}-\omega_{k}{}^{m}\,{}^{M}\mathbf{S}_{ml}{}^{\mu}-\omega_{l}{}^{m}\,{}^{M}\mathbf{S}_{km}{}^{\mu}-2t_{[k}{}^{M}\mathbf{T}_{l]}{}^{\mu}+\epsilon^{\mu}{}_{,\nu}{}^{M}\mathbf{S}_{kl}{}^{\nu}-\epsilon^{\lambda}{}_{,\lambda}{}^{M}\mathbf{S}_{kl}{}^{\mu}. \tag{3.21}$$

Equation (2.67) reduces to

$$^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu}=\delta_{\mu}{}^{\nu}\mathbf{L}^{T}-\mathbf{F}_{k}{}^{\lambda\nu}A^{k}{}_{\lambda,\mu}-\frac{\partial\mathbf{L}^{T}}{\partial\psi^{k}{}_{,\nu}}\psi^{k}{}_{,\mu} \tag{3.22}$$

with $\mathbf{L}^{T}\overset{\mathrm{def}}{=}\sqrt{-g}L^{T}$, while Eqs. (2.68)–(2.71) remain unchanged.

The densities $^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu}$ and $^{M}\mathbf{T}_{\mu}{}^{\nu}$ transform according to

$$^{G}\tilde{\mathbf{T}}'{}_{\mu}{}^{\nu}={}^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu}-t^{k}{}_{,\lambda\mu}\mathbf{F}_{k}{}^{\lambda\nu}+t^{k}{}_{,\mu}\frac{\partial\mathbf{L}^{T}}{\partial\psi^{k}{}_{,\nu}}+\omega^{kl}{}_{,\mu}\frac{\partial\mathbf{L}^{T}}{\partial\psi^{[k}{}_{,\nu}}\psi_{l]}+\epsilon^{\lambda}{}_{,\mu}{}^{G}\tilde{\mathbf{T}}_{\lambda}{}^{\nu}-\epsilon^{\nu}{}_{,\lambda}{}^{G}\tilde{\mathbf{T}}_{\mu}{}^{\lambda}-\epsilon^{\lambda}{}_{,\lambda}{}^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu}+\epsilon^{\rho}{}_{,\lambda\mu}\boldsymbol{\Psi}_{\rho}{}^{\lambda\nu}, \tag{3.23}$$

$$^{M}\mathbf{T}'{}_{\mu}{}^{\nu}={}^{M}\mathbf{T}_{\mu}{}^{\nu}+t^{k}{}_{,\mu}\frac{\partial\mathbf{L}^{M}}{\partial\psi^{k}{}_{,\nu}}+\omega^{kl}{}_{,\mu}\frac{\partial\mathbf{L}^{M}}{\partial\psi^{[k}{}_{,\nu}}\psi_{l]}+\epsilon^{\lambda}{}_{,\mu}{}^{M}\mathbf{T}_{\lambda}{}^{\nu}-\epsilon^{\nu}{}_{,\lambda}{}^{M}\mathbf{T}_{\mu}{}^{\lambda}-\epsilon^{\lambda}{}_{,\lambda}{}^{M}\mathbf{T}_{\mu}{}^{\nu}, \tag{3.24}$$

under the product transformations of Eqs. (2.22) and (2.23). The transformation properties of $^{G}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu}$ and of $^{M}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu}$ are given by the same forms as Eqs. (2.74) and (2.75), respectively, where, for the present case, $^{G}\tilde{\mathbf{T}}'{}_{\mu}{}^{\nu}$ and $^{M}\mathbf{T}'{}_{\mu}{}^{\nu}$ are given by Eqs. (3.23) and (3.24), respectively. Equation (2.76) is reduced to

$$\boldsymbol{\Psi}'{}_{\lambda}{}^{\mu\nu}=\boldsymbol{\Psi}_{\lambda}{}^{\mu\nu}+t^{k}{}_{,\lambda}\mathbf{F}_{k}{}^{\mu\nu}-\epsilon^{\rho}{}_{,\lambda}\boldsymbol{\Psi}_{\rho}{}^{\mu\nu}+\epsilon^{\mu}{}_{,\rho}\boldsymbol{\Psi}_{\lambda}{}^{\rho\nu}+\epsilon^{\nu}{}_{,\rho}\boldsymbol{\Psi}_{\lambda}{}^{\mu\rho}-\epsilon^{\rho}{}_{,\rho}\boldsymbol{\Psi}_{\lambda}{}^{\mu\nu}. \tag{3.25}$$

IV. SUMMARY AND DISCUSSIONS

We have examined the transformations properties of energy-momentum densities and of angular momentum densities both in PGT and in ENGR.

Results can be summarized as follows:

[1] Results in Poincaré gauge theory (PGT):

(1A) From Eqs. (2.63)–(2.66), we see that the densities $^{G}\mathbf{T}_{k}{}^{\mu}$, $^{M}\mathbf{T}_{k}{}^{\mu}$, $^{G}\mathbf{S}_{kl}{}^{\mu}$ and $^{M}\mathbf{S}_{kl}{}^{\mu}$ are all space-time vector densities, i.e., they transform as vector densities under general coordinate transformations. Their transformation properties under internal Poincaré transformations are summarized as follows:

(a) The dynamical energy-momentum density $^{G}\mathbf{T}_{k}{}^{\mu}$ of the gravitational field is invariant under *local* translations. It transforms as a vector under *global* $SL(2,C)$ transformations. But, it is not vectorial under *local* $SL(2,C)$ transformations.

(b) The dynamical energy-momentum density $^{M}\mathbf{T}_{k}{}^{\mu}$ of the matter field ϕ^{A} is invariant under *local* translations. It transforms as a vector under *local* $SL(2,C)$ transformations.

(c) Under *global* translations, the ''spin'' angular momentum density $^{G}\mathbf{S}_{kl}{}^{\mu}$ of the gravitational field receives transformations which correspond to translations in internal space-time, and it transforms as a tensor under *global* $SL(2,C)$ transformations. But, it is not tensorial under *local* Poincaré transformations.

(d) The ''spin'' angular momentum density $^{M}\mathbf{S}_{kl}{}^{\mu}$ of the matter field ϕ^{A} is tensorial under *local* Poincaré transformations.

(1B) From Eqs. (2.72)–(2.76), we see that $^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu}$, $^{M}\mathbf{T}_{\mu}{}^{\nu}$, $^{G}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu}$, and $^{M}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu}$ are all invariant under *global* internal Poincaré transformations. They are not invariant under *local* Poincaré transformations. Also, we can see the following:

(e) The canonical energy-momentum density $^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu}$ of the gravitational field transforms as tensor densities under affine coordinate transformation $x'^{\mu}=a^{\mu}{}_{\nu}x^{\nu}+b^{\mu}$, but it does not transform as a tensor density under general coordinate transformations [21].

(f) The canonical energy-momentum density $^{M}\mathbf{T}_{\mu}{}^{\nu}$ of the matter field ϕ^{A} transforms as a tensor density under general coordinate transformations.

(g) Both of ''extended orbital angular momentum'' densities $^{G}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu}$ and $^{M}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu}$ transform as tensor densities under constant $GL(4,R)$-coordinate transformations, and they receive space-time translations under constant coordinate transformations. They do not transform as tensor densities under general coordinate transformations.

[2] Results in extended new general relativity (ENGR):

(2A) All the densities $^{G}\mathbf{T}_{k}{}^{\mu}$, $^{M}\mathbf{T}_{k}{}^{\mu}$, $^{G}\mathbf{S}_{kl}{}^{\mu}$, and $^{M}\mathbf{S}_{kl}{}^{\mu}$ are space-time vector densities. Also for the case of ENGR, the same statements as (a), (b), and (d) in (1A) hold true for $^{G}\mathbf{T}_{k}{}^{\mu}$, $^{M}\mathbf{T}_{k}{}^{\mu}$, and $^{M}\mathbf{S}_{kl}{}^{\mu}$. As for $^{G}\mathbf{S}_{kl}{}^{\mu}$, we have the following [22]:

(c′) The density $^{G}\mathbf{S}_{kl}{}^{\mu}$ receives transformations which correspond to translations of the origin of internal space-time under *global* translations, and it transforms as a tensor under *local* $SL(2,C)$ transformations. But, it is not tensorial under *local* internal translations.

(2B) Also for $^{G}\tilde{\mathbf{T}}_{\mu}{}^{\nu}$, $^{M}\mathbf{T}_{\mu}{}^{\nu}$, $^{G}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu}$, and $^{M}\tilde{\mathbf{M}}_{\lambda}{}^{\mu\nu}$ in ENGR, the same statements as in (1B) hold true.

Since $^{G}\mathbf{T}_{k}{}^{\mu}$, $^{M}\mathbf{T}_{k}{}^{\mu}$, $^{G}\mathbf{S}_{kl}{}^{\mu}$, and $^{M}\mathbf{S}_{kl}{}^{\mu}$ are all space-time vector densities in both theories, the energy-momenta and angular momenta of the gravitational field and of the matter field are defined well, and independent of the coordinate system employed. For example, the energy-momentum $^{G}M_{k}$ of the gravitational field is defined by

$$ {}^{G}M_k \overset{\text{def}}{=} \int_\sigma {}^{G}\mathbf{T}_k{}^\mu d\sigma_\mu , \tag{4.1} $$

and we have

$$ {}^{G}M_k = \int_\sigma {}^{G}\mathbf{T}_k{}^\mu d\sigma_\mu = \int_\sigma {}^{G}\mathbf{T}'{}_k{}^\mu d\sigma'{}_\mu . \tag{4.2} $$

As we have mentioned in the final parts of Secs. II A and III A, the total energy-momentum and the *total* (=*spin* +*orbital*) angular momentum are given by M_k and S_{kl} for an asymptotically flat space-time, while the canonical energy-momentum $M^c{}_\mu$ and ''extended orbital angular momentum'' $L_\mu{}^\nu$, which are obtained as the integrations of nontensorial quantities $\tilde{\mathbf{T}}_\mu{}^\nu$ and $\tilde{\mathbf{M}}_\lambda{}^{\mu\nu}$, on the other hand, vanish and are trivial.

In both in PGT and in ENGR, the densities ${}^{M}\mathbf{T}_k{}^\mu$ and ${}^{M}\mathbf{S}_{kl}{}^\mu$ are well behaved under *local* internal Poincaré transformations, while the energy-momentum density ${}^{G}\mathbf{T}_k{}^\mu$ of the gravitational field is well behaved under *local* internal translations. In ENGR, the ''spin'' angular momentum density ${}^{G}\mathbf{S}_{kl}{}^\mu$ of the gravitational field is well behaved under *local* internal $SL(2,C)$ transformations.

It is worth mentioning here that the Lagrangian densities $L^G + L^M$ in PGT and $L^T + L^M$ in ENGR are both invariant under *local* internal translations and under *global* internal $SL(2,C)$ transformations, but they violate the invariance under *local* internal $SL(2,C)$ transformations. Thus, one may claim that we need not bother about the fact that ${}^{G}\mathbf{T}_k{}^\mu$ in PGT and in ENGR and ${}^{G}\mathbf{S}_{kl}{}^\mu$ in PGT are not tensorial under *local* internal $SL(2,C)$ transformations. In ENGR, in particular, this can be strongly asserted, because *local* $SL(2,C)$-gauge invariance is rather accidental [18] in this theory due to the lack of $SL(2,C)$-gauge potential $A^{kl}{}_\mu$.

We now give comments on alternative choices of sets of independent field variables:

{1} In PGT, we can choose $\{\psi^k, e^k{}_\mu, A^{kl}{}_\mu, \phi^A\}$ as the set of independent field variables [5,6,23]. The dynamical and canonical energy-momentum densities and spin and ''extended orbital angular momentum'' densities can be defined also for this case. The dynamical energy-momentum and spin angular momentum densities are space-time vector densities, and the canonical energy-momentum and ''extended orbital angular momentum'' densities are not space-time tensor densities. The transformation properties of these quantities under the Poincaré gauge transformations are much the same as in the case with $\{\psi^k, A^k{}_\mu, A^{kl}{}_\mu, \phi^A\}$ being employed. For the choice $\{\psi^k, e^k{}_\mu, A^{kl}{}_\mu, \phi^A\}$, however, the total dynamical energy-momentum vanishes identically and the total canonical energy-momentum gives the total energy-momentum for the asymptotically flat space-time for a suitably chosen coordinate system. Also, for asymptotically flat space-time, the spin angular momentum and orbital angular momentum are both divergent, and the total angular momentum is obtainable only as the sum of spin and orbital angular momenta. Thus, *the total energy-momentum and the total angular momentum cannot be defined independently of the coordinate system employed*, because both of the canonical energy-momentum and orbital angular momentum densities are not tensor densities.

{2} In ENGR, we can choose $\{\psi^k, e^k{}_\mu, \phi^A\}$ as the set of independent field variables [7,24]. For this choice, almost the same statements as in {1} hold, and the total energy-momentum and total angular momentum are obtained only by using densities which are not tensor densities.

The choice $\{\psi^k, A^k{}_\mu, A^{kl}{}_\mu, \phi^A\}$ with the condition (2.49) in PGT and the choice $\{\psi^k, A^k{}_\mu, \phi^A\}$ with the condition (2.49) in ENGR are preferential to all the other choices.

In general relativity, all the known energy-momentum and angular momentum densities of the gravitational field are not space-time tensor densities. Poincaré gauge theory and extended new general relativity are preferential to general relativity in the point that the former two theories have energy-momentum and angular momentum densities of the gravitational field which are true space-time vector densities [25].

APPENDIX

The irreducible components t_{klm}, v_k, and a_k of T_{klm} are defined by the following:

$$ t_{klm} \overset{\text{def}}{=} \frac{1}{2}(T_{klm}+T_{lkm}) + \frac{1}{6}(\eta_{mk}v_l + \eta_{ml}v_k) - \frac{1}{3}\eta_{kl}v_m , \tag{A1} $$

$$ v_k \overset{\text{def}}{=} T^l{}_{lk} , \tag{A2} $$

$$ a_k \overset{\text{def}}{=} \frac{1}{6}\varepsilon_{klmn}T^{lmn}, \tag{A3} $$

where the symbol ε_{klmn} stands for completely antisymmetric Lorentz tensor with $\varepsilon_{(0)(1)(2)(3)} = -1$ [26].

The irreducible components $A_{klmn}, B_{klmn}, C_{klmn}, E_{kl}, I_{kl}, R$ of R_{klmn} are defined by the following:

$$ A_{klmn} \overset{\text{def}}{=} \frac{1}{6}(R_{klmn}+R_{kmnl}+R_{knlm}+R_{lmkn}+R_{lnmk}+R_{mnkl}), \tag{A4} $$

$$ B_{klmn} \overset{\text{def}}{=} \frac{1}{4}(W_{klmn}+W_{mnkl}-W_{knlm}-W_{lmkn}), \tag{A5} $$

$$ C_{klmn} \overset{\text{def}}{=} \frac{1}{2}(W_{klmn}-W_{mnkl}), \tag{A6} $$

$$ E_{kl} \overset{\text{def}}{=} \frac{1}{2}(R_{kl}-R_{lk}), \tag{A7} $$

$$ I_{kl} \overset{\text{def}}{=} \frac{1}{2}(R_{kl}+R_{lk}) - \frac{1}{4}\eta_{kl}R, \tag{A8} $$

$$ R = \eta^{kl}R_{kl} \tag{A9} $$

with

$$ W_{klmn} \overset{\text{def}}{=} R_{klmn} - \frac{1}{2}(\eta_{km}R_{ln} + \eta_{ln}R_{km} - \eta_{kn}R_{lm} - \eta_{lm}R_{kn}) + \frac{1}{6}(\eta_{km}\eta_{ln} - \eta_{lm}\eta_{kn})R, \tag{A10} $$

$$ R_{kl} \overset{\text{def}}{=} \eta^{mn}R_{kmln} . \tag{A11} $$

[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), p. 466.
[2] T. Kawai, Gen. Relativ. Gravit. **18**, 995 (1986); **19**, 1285(E) (1987).
[3] It is remarkable that the generator S_{kl} of internal $SL(2,C)$ transformations gives the *total* (= *spin* + *orbital*) angular momentum. The word ''spin'' is to describe this situation.
[4] T. Kawai, Prog. Theor. Phys. **79**, 920 (1988).
[5] T. Kawai and H. Saitoh, Prog. Theor. Phys. **81**, 280 (1989).
[6] T. Kawai and H. Saitoh, Prog. Theor. Phys. **81**, 1119 (1989).
[7] T. Kawai and N. Toma, Prog. Theor. Phys. **85**, 901 (1991).
[8] The covering group $\overline{P_0}$ of the proper orthochronous Poincaré group will be called the $\overline{\text{Poincaré}}$ group, and its elements will be called $\overline{\text{Poincaré}}$ transformations.
[9] Unless otherwise stated, we use the following conventions for indices: The middle part of the Greek alphabet, $\mu,\nu,\lambda, \ldots$, refers to 0, 1, 2, and 3. In a similar way, the middle part of the Latin alphabet, $i,j,k, \ldots$, means 0, 1, 2, and 3, unless otherwise stated. The capital letters A and B are used for indices for components of the field ϕ, and N denotes the dimension of the representation ρ.
[10] For function f on M, we define $f_{,\mu} \stackrel{\text{def}}{=} \partial f/\partial x^{\mu}$.
[11] We define

$$A_{\cdots[\mu\cdots\nu]\cdots} \stackrel{\text{def}}{=} \frac{1}{2}(A_{\cdots\mu\cdots\nu\cdots} - A_{\cdots\nu\cdots\mu\cdots}),$$

$$A_{\cdots(\mu\cdots\nu)\cdots} \stackrel{\text{def}}{=} \frac{1}{2}(A_{\cdots\mu\cdots\nu\cdots} + A_{\cdots\nu\cdots\mu\cdots}).$$

[12] T. Kawai, Prog. Theor. Phys. **82**, 850 (1989).
[13] The field components $e^k{}_\mu$ and $e^\mu{}_k$ are used to convert Latin and Greek indices, similarly as in the case of $D_k\phi^A$ and $D_\mu\phi^A$. Also, raising and lowering the indices $k,l,m, \ldots$ are accomplished with the aid of $(\eta^{kl}) \stackrel{\text{def}}{=} (\eta_{kl})^{-1}$ and (η_{kl}).
[14] K. Hayashi and T. Shirafuji, Prog. Theor. Phys. **64**, 866 (1980); **64**, 883 (1980); **64**, 1435 (1980).
[15] The requirement that the theory has the Newtonian limit restricts the parameter d as $d=1/(2\kappa)$ [14]. Here, κ denotes the Einstein constant $\kappa=8\pi G/c^4$ with G and c being the Newton gravitational constant and the light velocity in the vacuum, respectively. If $3d_2+2d_3=0=d_5+12d_6$ and if the intrinsic spins of the source fields are negligible in addition, the Einstein equation is obtainable [14]. In the discussions below, however, we do not impose any restriction on the parameters c_i, d_j, d $(i=1,2,3, j=1,2,3,\cdots,6)$.
[16] Note that the antisymmetric part $L_{[\mu\nu]}$ is the generator of coordinate Lorentz transformations and is orbital angular momentum.
[17] K. Hayashi and T. Shirafuji, Phys. Rev. D **19**, 3524 (1979).
[18] The case with $c_1=-1/(3\kappa)=-c_2$ is quite favorable experimentally. Further, when $c_3=-3/(4\kappa)$ in addition, the Lagrangian density L^T agrees with $R(\{\})/(2\kappa)$ up to a total divergence, and the gravitational field equation agrees with the Einstein equation [17]. Here, $R(\{\})$ denotes the Riemann scalar curvature. Even for this case, L^T is not invariant under *local* $SL(2,C)$-gauge transformations, while $R(\{\})/(2\kappa)$ is. The discussions below will be developed for arbitrary values of the parameters c_1, c_2, and c_3.
[19] In the rest of this section, the set $\{\psi^k, A^k{}_\mu, \phi^A\}$ will be employed as the set of independent field variables. The case when $\{\psi^k, e^k{}_\mu, \phi^A\}$ is employed will be mentioned in the final section.
[20] For the case with $c_1=-1/(3\kappa)=-c_2, c_3=-3/(4\kappa)$ in ENGR, the gravitational energy-momentum density ${}^G\mathbf{T}_k{}^\mu$ reduces essentially to $hj_a{}^\rho$ introduced in V. C. de Andrade, L. C. T. Guillen, and J. G. Pereira, Phys. Rev. Lett. **84**, 4533 (2000).
[21] The symbol $\tilde{}$ in the symbols ${}^G\tilde{\mathbf{T}}_\mu{}^\nu$, ${}^G\tilde{\mathbf{M}}_\lambda{}^{\mu\nu}$, etc. is to express that these densities are not space-time tensor densities.
[22] Note that ${}^G\mathbf{S}_{kl}{}^\mu$ in PGT is *not* tensorial under *local* $SL(2,C)$ transformations, as has been stated in (c).
[23] This choice corresponds to considering PGT. As for the total energy-momentum and the total angular momentum in PGT, see K. Hayashi and T. Shirafuji, Prog. Theor. Phys. **73**, 54 (1985).
[24] This choice corresponds to considering NGR.
[25] Please recall the comments [15,18].
[26] Latin indices are put in parentheses to distinguish them from Greek indices.